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ON THE TWISTED ALEXANDER POLYNOMIAL FOR METABELIAN $\mathrm{SL}_2(\mathbb{C})$-REPRESENTATIONS WITH THE ADJOINT ACTION

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1. Introduction

We devote this note to expose an explicit form of the twisted Alexander invariant for irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations of knot groups. This work was motivated by the characterization of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations in [9], concerning the conjugacy classes of $\mathrm{SL}_2(\mathbb{C})$-representations. We can correspond the set of conjugacy classes of $\mathrm{SL}_2(\mathbb{C})$-representations to an affine variety called the *character variety* (for details, we refer to [2, 7]). The conjugacy classes of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations forms the fixed points on the character variety under an involution ($\mathbb{Z}_2$-action). Since the twisted Alexander invariant has the invariance under conjugation of representations, it is expected that the feature of conjugacy classes of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations is carried over into the computation result of the twisted Alexander invariant for irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations. In particular, we consider the composition of $\mathrm{SL}_2(\mathbb{C})$-representations with the adjoint action. Since the adjoint action connects the homology of group with the cotangent space on the character variety, we can expect that the twisted Alexander invariant have a more significant feature concerning the linear map induced by the involution on the cotangent space at a fixed point. Our main theorem is stated as follows:

Main Theorem *If an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(E_K)$ is metabelian and longitude-regular (requiring irreducibility and some additional conditions), then the twisted Alexander invariant for the composition of ρ with the adjoint action factors into the product*

$$(t-1)\Delta_K(-t)P(t)$$

where $\Delta_K(t)$ is the Alexander polynomial of K and $P(t)$ is a Laurent polynomial satisfying that $P(t)=P(-t)$.

Throughout this note, we use the symbol K for a knot in S^3 and E_K for the knot exterior $S^3 \setminus N(K)$ where $N(K)$ is an open tubular neighbourhood of K. Hence $\pi_1(E_K)$ denotes the knot group of K.

In the Main theorem, it seems that the symmetry of $P(t)$ corresponds to the feature of the conjugacy class of ρ as a fixed point under the involution and the Alexander polynomial with the variable multiplied with -1 seems to be the effect by the linear map induced by the involution on the cotangent space at the fixed point.

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We aim to observe the twisted Alexander invariant for the composition of irreducible metabelian $SL_2(\mathbb{C})$-representations with the adjoint action (for the definition, see Section 2) and compute concrete examples. For this purpose, we need a pair of a suitable presentations of knot groups and explicit forms of irreducible metabelian $SL_2(\mathbb{C})$-representations. X.-S. Lin [6] has introduced such a useful presentation of knot groups by using free Seifert surfaces for knots.

Instead of giving the rigorous proof to our main theorem, we discuss the details of construction and computation for Lin's special presentations of knot groups and show computation procedures of the twisted Alexander invariant via concrete examples.

Organization

First we will review the twisted Alexander invariant for the composition of $SL_2(\mathbb{C})$-representations with the adjoint action in Section 2. Section 3 shows a brief exposition of metabelian $SL_2(\mathbb{C})$-representations of knot groups and its characterization in the character variety. Section 4 gives a review on special presentations of knot groups, by using free Seifert surfaces, and the detail on how to write down such presentations via the concrete example for the trefoil knot. In Section 5, we will state our main theorem on the twisted Alexander invariant for the composition of irreducible metabelian $SL_2(\mathbb{C})$-representations with the adjoint action and the sketch of the proof. Last, we calculate the twisted Alexander invariants of the trefoil knot, figure eight knot and 5_2 knot for the composition of irreducible metabelian $SL_2(\mathbb{C})$-representations with the adjoint action in Section 6.

2. Review of the twisted Alexander invariant

We review the definition of twisted Alexander invariant. We follow the definition in the way of Wada [12] by using Fox differential calculus on knot groups. To define the twisted Alexander invariant, we need a presentation and two homomorphisms of a knot group.

One homomorphism is the abelianization homomorphism of a knot group. The abelianization homomorphism is the quotient one by the commutator subgroup and the quotient group is called the abelianization of a group. It is known that the abelianization of a fundamental group is isomorphic to the first homology group. Since the abelianization of a knot group is a free abelian group with rank one, we express this abelianization as the multiplicative group $\langle t \rangle$. We denote by α the following abelianization of $\pi_1(E_K)$:

$$\pi_1(E_K) \to \langle t \rangle, \quad \mu \mapsto t$$

where μ is a meridian of the knot K. The other homomorphism is called a *representation* of a knot group. Representations means homomorphisms from a group into a linear automorphism group of a vector space. In this note, we consider representations into $SL_2(\mathbb{C})$, i.e., a representation ρ is a homomorphism from $\pi_1(E_K)$ into $SL_2(\mathbb{C})$ and we take the composition of an $SL_2(\mathbb{C})$-representation with the adjoint action.

Definition 2.1. The Lie group $SL_2(\mathbb{C})$ acts on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by conjugation:

$$\begin{aligned} A : \mathfrak{sl}_2(\mathbb{C}) &\to \mathfrak{sl}_2(\mathbb{C}) \\ v &\mapsto AvA^{-1} \end{aligned}$$

where $A \in \mathrm{SL}_2(\mathbb{C})$. This is called the *adjoint action* of A and denoted by the symbol Ad_A.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the following three matrices over $\mathbb{C}$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

In particular, when we regard $\mathfrak{sl}_2(\mathbb{C})$ as a 3-dimensional vector space over $\mathbb{C}$, the adjoint action turns into a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ into $Aut(\mathfrak{sl}_2(\mathbb{C})) \simeq Aut(\mathbb{C}^3)$. It is also known that the determinant of the adjoint action is always 1. More precisely if an element $A \in \mathrm{SL}_2(\mathbb{C})$ has the eigenvalues $\xi^{\pm 1}$, then the composition Ad_A has the eigenvalues $\xi^{\pm 2}$ and 1 (see Eq. (6) for example). Hence the composition of an $\mathrm{SL}_2(\mathbb{C})$-representation ρ with the adjoint action gives an $\mathrm{SL}_3(\mathbb{C})$-representation of $\pi_1(E_K)$:

$$Ad \circ \rho : \pi_1(E_K) \xrightarrow{\rho} \mathrm{SL}_2(\mathbb{C}) \xrightarrow{Ad} Aut(\mathfrak{sl}_2(\mathbb{C})).$$

These compositions with the adjoint action appear homology of groups with coefficient in $\mathfrak{sl}_2(\mathbb{C})$ (we refer to [10] and [11, Lecture 15] for SU(2) case).

We also review the definition of the twisted Alexander invariant for the composition of an $\mathrm{SL}_2(\mathbb{C})$-representation ρ of a knot group $\pi_1(E_K)$ with the adjoint action.

Definition 2.2. We choose a presentation of a knot group $\pi_1(E_K)$ as

$$\pi_1(E_K) = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_{k-1} \rangle$$

and an $\mathrm{SL}_2(\mathbb{C})$-representation ρ. Let $\Phi_{Ad\circ\rho}$ be the linear extension of the tensor product $\alpha \otimes Ad_\rho\colon \pi_1(E_K) \to \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} \mathrm{SL}_3(\mathbb{C})$ on the group ring $\mathbb{Z}[\pi_1(E_K)]$, i.e.,

$$\begin{aligned} \Phi_{Ad\circ\rho}\colon \mathbb{Z}[\pi_1(E_K)] &\to \mathbb{C}[t^{\pm 1}] \otimes \mathrm{M}_3(\mathbb{C}) = \mathrm{M}_3(\mathbb{C}[t^{\pm 1}]) \\ \sum_i a_i \gamma_i &\mapsto \sum_i a_i \alpha(\gamma_i) \otimes Ad \circ \rho(\gamma_i) \end{aligned}$$

Here we identify $\mathbb{C}[t^{\pm 1}] \otimes \mathrm{M}_3(\mathbb{C})$ with $\mathrm{M}_3(\mathbb{C}[t^{\pm 1}])$. We assume that $\alpha(g_1) \neq 1$. Then the twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$ is defined as the following ratio of two determinants of elements in $\mathrm{M}_3(\mathbb{C}[t^{\pm 1}])$:

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) = \frac{\det\left(\Phi_{Ad\circ\rho}\left(\dfrac{\partial r_i}{\partial g_j}\right)\right)_{\substack{1\leqq i\leqq k-1,\\ 2\leqq j\leqq k}}}{\det(\Phi_{Ad\circ\rho}(g_1 - 1))}. \tag{2}$$

Remark 2.3. When we consider the rational function

$$\frac{\det\left(\Phi_{Ad\circ\rho}\left(\dfrac{\partial r_i}{\partial g_j}\right)\right)_{\substack{1\leqq i\leqq k-1,\\ 1\leqq j\leqq k, j\neq \ell}}}{\det(\Phi_{Ad\circ\rho}(g_\ell - 1))}.$$

for other generator g_ℓ satisfying that $\alpha(g_\ell) \neq 1$, we have the same rational function as Eq. (2) up to a factor $\pm t^n$ ($n \in \mathbb{Z}$). In this note, we choose the last generator in a presentation of a knot group for our concrete examples in Section 6.

3. METABELIAN REPRESENTATIONS

We mainly consider the special $SL_2(\mathbb{C})$-representations, which are called *metabelian.* In particular, we focus on *irreducible* metabelian $SL_2(\mathbb{C})$-representations in this note.

Definition 3.1. An $SL_2(\mathbb{C})$-representation ρ of $\pi_1(E_K)$ is metabelian if the image of the commutator subgroup $[\pi_1(E_K), \pi_1(E_K)]$ by ρ is an abelian subgroup in $SL_2(\mathbb{C})$.

In the definition 3.1, we consider the condition concerning the image of the comutator sugroup by an $SL_2(\mathbb{C})$-representation. Concerning the whole image of $\pi_1(E_K)$, we often consider the existence on a common eigenspace for all $SL_2(\mathbb{C})$-elements in the image of $\pi_1(E_K)$. According to the existence on a common eigenspace, an $SL_2(\mathbb{C})$-representation is referred to as being either *reducible* or *irreducible.*

Definition 3.2. An $SL_2(\mathbb{C})$-representation ρ is *reducible* if there exists an invariant line L in $\mathbb{C}^2$ such that $\rho(\gamma)(L) \subset L$ for all $\gamma \in \pi_1(E_K)$. This means that there exists a common eigenvector of $\rho(\gamma)$ for all $\gamma \in \pi_1(E_K)$. Hence by taking conjugate we can assume the image of $\pi_1(E_K)$ by a reducible $SL_2(\mathbb{C})$-representation is contained in upper triangular matrices in $SL_2(\mathbb{C})$. We call an $SL_2(\mathbb{C})$-representation ρ *irreducible* if ρ is not reducible.

Remark 3.3. By direct computation, for upper triangular $SL_2(\mathbb{C})$-matrices A and B we have

$$[A, B] = ABA^{-1}B^{-1} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

Together with the fact that all upper triangular matrices with diagonal components 1 forms an abelian subgroup in $SL_2(\mathbb{C})$, this means that all reducible representations are metabelian.

The twisted Alexander invariant for reducible $SL_2(\mathbb{C})$-representations is calculated explicitly, in [5, 14]. Therefore we focus on irreducible metabelian $SL_2(\mathbb{C})$-representations of $\pi_1(E_K)$ in the subsequent sections. For the exposition on the twisted Alexander invariant for metabelian $SL_2(\mathbb{C})$-representations, we refer to [15].

We deal with $SL_2(\mathbb{C})$-representations in the difference between reducible ones and metabelian ones. Such the difference is expressed as only finite number of conjugacy classes.

Remark 3.4. It has been shown in [6, 8] that the conjugacy classes of irreducible metabelian $SL_2(\mathbb{C})$-representations of $\pi_1(E_K)$ is finite and the number is given by

$$\frac{|\Delta_K(-1)| - 1}{2}$$

where $\Delta_K(t)$ the Alexander polynomial of K. For explicit forms of irreducible metabelian $SL_2(\mathbb{C})$-representations, see Proposition 5.3.

To characterize these conjugacy classes, we define an involution on the set of $SL_2(\mathbb{C})$-representations of a knot group by using scalar multiplication for matrices. For ρ is an $SL_2(\mathbb{C})$-representation of $\pi_1(E_K)$, we can define a new $SL_2(\mathbb{C})$-representation $(-1)^{[\cdot]}\rho$ as

$$\begin{aligned} (-1)^{[\cdot]}\rho \colon \pi_1(E_K) &\to SL_2(\mathbb{C}) \\ \gamma &\mapsto (-1)^{[\gamma]}\rho(\gamma) \end{aligned}$$

where $[\gamma]$ is the homology class of γ in $H_1(E_K;\mathbb{Z}) \simeq \mathbb{Z}$. It is easy to see that the correspondence $\rho \mapsto (-1)^{[\cdot]}\rho$ is an involution and induces the involution on the set of conjugacy classes.

Remark 3.5. It is shown in [9] that every irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representation ρ of $\pi_1(E_K)$ is conjugate to $(-1)^{[\cdot]}\rho$. Moreover it is also shown that an irreducible $\mathrm{SL}_2(\mathbb{C})$-representation ρ is metabelian if it is conjugate to $(-1)^{[\cdot]}\rho$. This means that the conjugacy classes of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations form the fixed points in the $\mathrm{SL}_2(\mathbb{C})$-character variety of $\pi_1(E_K)$ under the involution.

Remark 3.6. The higher rank analog ($\mathrm{SL}_n(\mathbb{C})$ cases) in Remark 3.5 is given by H. Boden and S. Friedl in [1].

We can expect that the invariance of irreducible metabelian representation under the action of $\mathbb{Z}_2$ gives rise to significant features of the twisted Alexander invariant for irreducible metabelian representations. For the computation procedure of the twisted Alexander invariant, we need a suitable presentation of a knot group to write down irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations explicitly.

4. Review of Lin presentations

To investigate metabelian $\mathrm{SL}_2(\mathbb{C})$-representations, it is useful to use the special presentations of knot groups, introduced by X.-S. Lin in [6]. We call such presentations *Lin presentations* of $\pi_1(E_K)$. We review the definition of Lin presentations and show how to obtain such presentation of $\pi_1(E_K)$ with an explicit example.

4.1. Definition of Lin presentations.

In the definition of Lin presentations, we need *free* Seifert surfaces of knots. We start with the definition of free Seifert surfaces.

Definition 4.1. A Seifert surface of a knot is *free* if the complement of an open tubular neighbourhood of S in S^3 is a handlebody. Hence $\pi_1(S^3 \setminus N(S))$ is a free group with rank $2g$ where $N(S)$ is an open tubular neighbourhood of S and g is the genus of S.

For example, we can see a free Seifert surface of the trefoil knot as in Figure 1. To see

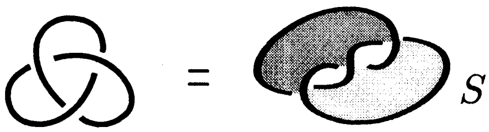

FIGURE 1. A free Seifert surface S of the trefoil knot

that the Seifert surface as in Figure 1 is free, we make a Heegaard splitting of S^3 by using the Seifert surface along the following procedure:

1. Decompose S^3 into the union $B_1 \cup B_2$ of two 3-balls where B_1 contains the Seifert surface S as the left side in Figure 2.
2. Remove two 1-handles along the loops x_1 and x_2 outside the Seifert surface S from B_1 and attach these two 1-handles to B_2 as the right side in Figure 2.

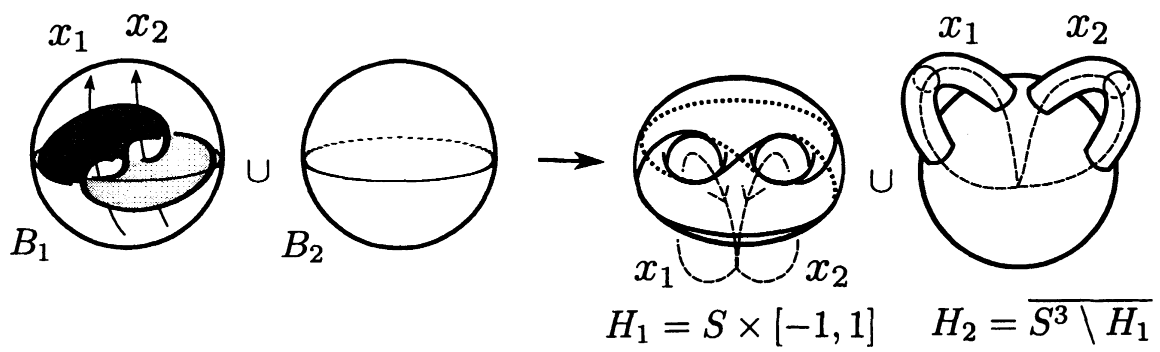

FIGURE 2. Heegaard decomposition by the free Seifert surface of the trefoil knot

We define a Lin presentation of $\pi_1(E_K)$ associated with a free Seifert surface S of a knot K. The generators consist of the generators $x_1, \dots, x_{2g}$ of $\pi_1(S^3 \setminus N(S))$ and a meridian μ. The relations are given by $2g$ loops in the spine of S. Here the spine of a Seifert surface is a deformation retract of the Seifert surface. That deformation retract is given by a bouquet of circles $a_1, \dots, a_{2g}$ since a Seifert surface is a compact connected surface with one boundary circle. The homotopy class of the loop a_i^+ (resp. a_i^-), given by pushing up (resp. down) the loop a_i, is expressed as a word in $x_1 \dots, x_{2g}$. One can see the relation $\mu a_i^+ \mu^{-1} = a_i^-$ for these two words a_i^+ and a_i^-. We have a presentation which consists of $2g+1$ generators and $2g$ relations as follows.

Definition 4.2. We choose a free Seifert surface S of a knot K. When we denote by $x_1, \dots, x_{2g}$ the generators of the free group $\pi_1(S^3 \setminus N(S))$, we can express the knot group as

$$\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \,|\, \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$$

where $a_i^\pm$ are words in $x_1, \dots, x_{2g}$ and denote the homotopy classes of loops given by pushing up and down the loop a_i in the spine $\vee a_i$ of S. We call this presentation a *Lin presentation* associated with S.

4.2. **How to compute relations in Lin presentations.** In this section, we describe relations of Lin presentations in details via the trefoil knot. To obtain relations of a Lin presentation associated with a free Seifert surface S, it is enough to write down the loops $a_i^\pm$ given by pushing up and down a_i in the spine of S as element in $\pi_1(S^3 \setminus N(S))$. Hence by chasing the intersection of $a_i^\pm$ with the cocores of 1-handles in the handlebody $S^3 \setminus N(S)$, we can describe the homotopy classes of $a_i^\pm$ as words in the generators of $\pi_1(S^3 \setminus N(S))$. We denote by x_i the generator in $\pi_1(S^3 \setminus N(S))$ corresponding to a 1-handle in $S^3 \setminus N(S)$ and by D_i the cocore of the 1-handle as in Figure 3. We set the orientations x_i and D_i such that the intersection is positive.

Lemma 4.3. *We suppose that a loop γ in $S^3 \setminus N(S)$ intersects with $D_{j_1}, D_{j_2}, \dots$ in this order. When we denote by $\epsilon_k \in \{\pm 1\}$ the sign of the intersection of γ with the disk D_{j_k}, the homotopy class of γ is given by the word $x_{j_1}^{\epsilon_1} x_{j_2}^{\epsilon_2} \cdots$.*

Example 4.4. The example of the trefoil knot. For the Seifert surface S in Figure 1, the spine of S is given by the bouquet $S^1 \vee S^1$ as in Figure 4.

By pushing up and down this spine $a_1 \vee a_2$, we have the closed loops a_1^+, a_2^+, a_1^- and a_2^- in the complement of the Seifert surface S as in Figures 5 & 6.

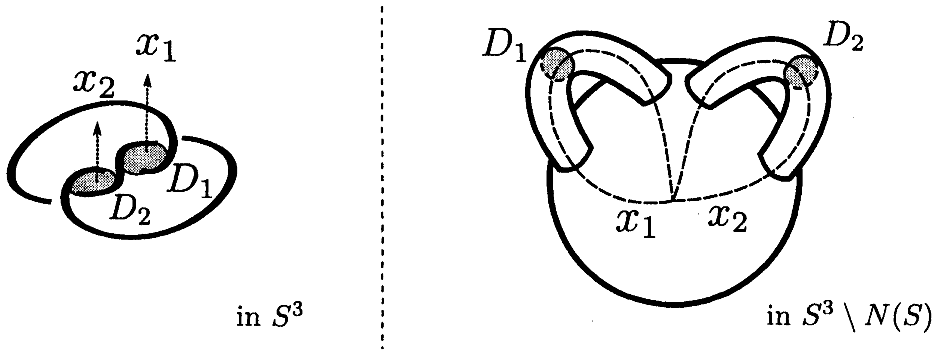

FIGURE 3. The cocores in 1-handles

FIGURE 4. The spine of Seifert surface for the trefoil knot

The fundamental group $\pi_1(S^3 \setminus N(S))$ is a free group and generated by the homotopy classes of x_1 and x_2. The homotopy classes of the closed loops $a_1^{\pm}$ and $a_2^{\pm}$ are expressed as words in x_1 and x_2. One can find that

(3) $$a_1^+ = x_1, \qquad a_1^- = x_1 x_2^{-1},$$

(4) $$a_2^+ = x_2^{-1} x_1, \qquad a_2^- = x_2^{-1}$$

where we use the same symbols for the homotopy classes of $a_i^{\pm}$ $(i = 1, 2)$ for simplicity.

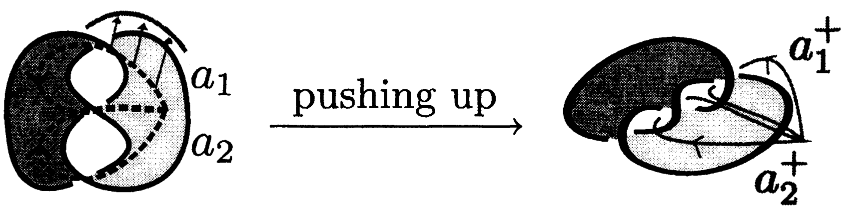

FIGURE 5. The loops a_1^+ and a_2^+ obtained by pushing up the spine

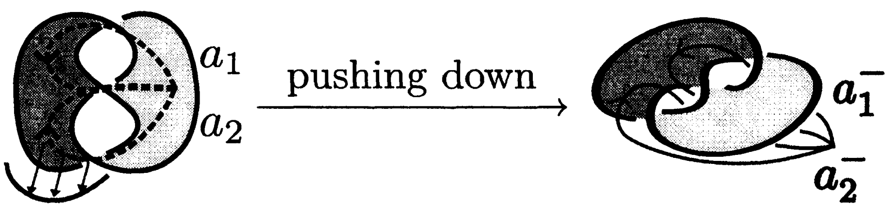

FIGURE 6. The loops a_1^- and a_2^- obtained by pushing down the spine

We deduce the above relations in (3) & (4) from counting the intersection of the closed loops $a_1^{\pm}$ and $a_2^{\pm}$ with the cocores D_1 and D_2 in the handlebody of $S^3 \backslash N(S)$ as in Figure 7. The closed loop a_1^{+} has the only positive intersection with D_1. The closed loop a_1^{-} has one positive intersection with D_1 and one negative intersection with D_2 in this order. Hence we also see the expressions $a_1^{+} = x_1$ and $a_1^{-} = x_1 x_2^{-1}$ in Eq. (3) by Lemma 4.3. We can see the expression in Eq. (4) similarly.

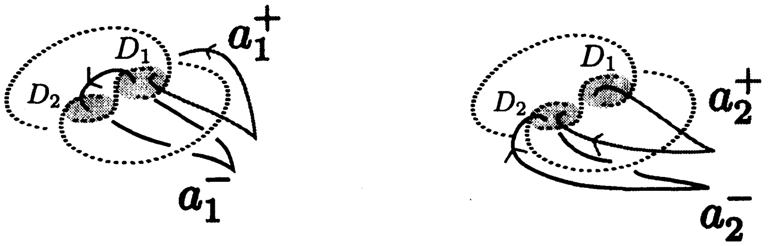

FIGURE 7. The intersections a_1^- and a_2^- with D_1 and D_2

We also see how the relations $\mu a_i^+ \mu^{-1} = a_i^-$ is illustrated for the trefoil knot. For example, the closed loops $a_1^{\pm}$ are obtained by pushing up and down the spine a_1 along the normal direction of the Seifert surface S as in Figures 5 & 6. Hence we put an annulus between a_1^+ and a_1^-. This annulus intersects with the trefoil knot at the two points as in Figure 8. By attaching two meridians to avoid the intersection points of the annulus with the trefoil knot, we can see the disk whose boundary is homotopic to the closed loop $\mu a_1^+ \mu^{-1} (a_1^-)^{-1}$.

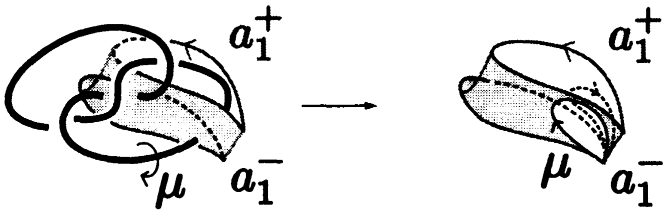

FIGURE 8. The homotopy between a_1^+ and a_1^-

5. Main theorem

In this section, We state the explicit form of the twisted Alexander invariant for the composition of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations with the adjoint action and give a sketch of the proof. In our theorem, we require a little more strong technical condition for metabelian representations than irreducibility. This condition is called *longitude–regular.* The irreducibility of representations is included in longitude–regularity (for details about the longitude–regularity, we refer to [15]). Our main theorem is stated as follows.

Theorem 5.1. *Let ρ be an $\mathrm{SL}_2(\mathbb{C})$-representation of a knot group $\pi_1(E_K)$. If ρ is metabelian and longitude–regular, then the twisted Alexander invariant $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ is expressed as*

$$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = (t-1)\Delta_K(-t)P(t)$$

where $\Delta_K(t)$ is the Alexander polynomial of K and $P(t)$ is a Laurent polynomial satisfying that $P(t) = P(-t)$.

Remark 5.2. Note that the assumption of longitude–regularity is a sufficient condition for the twisted Alexander invariant to be a Laurent polynomial.

To compute the twisted Alexander invariant, we need explicit forms of irreducible $\mathrm{SL}_2(\mathbb{C})$-representations. It is shown by using a Lin presentation of a knot group in [8] that we have the following representative in each conjugacy class of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations.

Proposition 5.3 (See the proof of Proposition 1.1 and Theorem 1.2 in [8]). *We suppose that a knot group $\pi_1(E_K)$ has a Lin presentation $\langle x_1, \ldots, x_{2g}, \mu \,|\, \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \ldots, 2g\rangle$. If ρ is an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representation, then ρ is conjugate to the $\mathrm{SL}_2(\mathbb{C})$-representation given by the following correspondence:*

$$\mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_i \mapsto \begin{pmatrix} \xi_i & 0 \\ 0 & \xi_i^{-1} \end{pmatrix} \quad (i = 1, \ldots, 2g) \tag{5}$$

where every ξ_i is a root of unity.

The twisted Alexander invariant has the invariance under conjugation of representations. Hereafter we consider irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations which sends the generators in a Lin presentation to the matrices as in Proposition 5.3. By direct calculation, we also obtain the following explicit forms of the composition of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations with the adjoint action.

Proposition 5.4. *Let ρ be an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representation of a knot group $\pi_1(E_K)$. We suppose that the knot group $\pi_1(E_K)$ has a Lin presentation*

$$\pi_1(E_K) = \langle x_1, \ldots, x_{2g}, \mu \,|\, \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \ldots, 2g\rangle$$

and ρ sends the generators $x_1, \ldots, x_{2g}$ and μ to the diagonal matrices and the trace-free matrix as in Eq. (5).

Then the composition of ρ with the adjoint action is decomposed into a direct sum of the following 1-dimensional representation ψ_1 and 2-dimensional representation ψ_2 of $\pi_1(E_K)$:

$$Ad \circ \rho = \psi_1 \oplus \psi_2$$

where ψ_1 is a $\mathrm{GL}_1(\mathbb{C})$-representation and ψ_2 is a $\mathrm{GL}_2(\mathbb{C})$-representation, given by the following correspondence:

$$\psi_1(\mu) = -1, \qquad \psi_1(x_i) = 1 \qquad (i = 1, \ldots, 2g),$$
$$\psi_2(\mu) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \psi_1(x_i) = \begin{pmatrix} \xi_i^2 & 0 \\ 0 & \xi_i^{-2} \end{pmatrix} \qquad (i = 1, \ldots, 2g).$$

Remark 5.5. The representations ψ_1 and ψ_2 are the restrictions of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representation ρ on the subspace $V_1 = \langle H\rangle$ and $V_2 = \langle E, F\rangle$ in $\mathfrak{sl}_2(\mathbb{C})$.

The proof of our main theorem is based on Proposition 5.4. We sketch the proof the main theorem.

A sketch of the proof. By Proposition 5.4 and the multiplicativity of the twisted Alexander invariant (we refer to [5]), we factor the twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$ into the product of two twisted Alexander invariants $\Delta_{E_K}^{\alpha\otimes\psi_1}(t)$ and $\Delta_{E_K}^{\alpha\otimes\psi_2}(t)$.

By the computation for 1-dimensional representations in [5, Section 3.3 Examples and computations of the twisted polynomials], the twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes\psi_1}(t)$ turns into the rational function $\Delta_K(-t)/(-t-1)$. On the other hand, by Wada's criterion [12, Proposition 8], twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes\psi_2}(t)$ turns into a Laurent polynomial $Q(t)$. Moreover by the invariance of the twisted Alexander invariant under conjugation, one can see that $Q(t) = Q(-t)$ via conjugation by the diagonal matrix $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Summarized the above, the twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$ turns into the following product:

$$\begin{aligned}\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) &= \Delta_{E_K}^{\alpha\otimes\psi_1}(t)\cdot\Delta_{E_K}^{\alpha\otimes\psi_2}(t)\\ &= \frac{\Delta_K(-t)}{-t-1}\cdot Q(t).\end{aligned}$$

Since we assume that ρ is longitude–regular, it follows from [13] that the twisted Alexander invariant $\Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t)$ has zero at $t=1$. It is known that $\Delta_K(-1)$ is an odd integer. Hence we factor $Q(t)$ into the product $(t-1)(t+1)P(t)$ by the symmetry that $Q(t)=Q(-t)$. This completes our proof. □

Remark 5.6. The factors $\Delta_K(-t)$ and $P(t)$ imply the features of conjugacy classes of irreducible metabelian representations in the character variety. The points corresponding to the conjugacy classes of irreducible metabelian representations forms the fixed points of the character variety under an action of $\mathbb{Z}_2$. The symmetry that $P(t)=P(-t)$ implies the invariance of conjugacy classes under $\mathbb{Z}_2$-action as the fixed points. The Alexander polynomial with the variable $-t$ seems to be related to the linear action on the cotangent spaces at the fixed points induced by $\mathbb{Z}_2$-action.

6. Examples

This section shows three concrete examples of the twisted Alexander invariant for the composition of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations with the adjoint action.

6.1. The trefoil knot.

We start with the trefoil knot and irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations of the knot group. We use the Lin presentation associated with the free Seifert surface as in Figure 1. Recall that the Lin presentation is expressed as

$$\pi_1(E_K) = \langle x_1, x_2, \mu \,|\, \mu x_1\mu^{-1} = x_1x_2^{-1}, \mu x_2^{-1}x_1\mu^{-1} = x_2^{-1}\rangle.$$

The number of conjugacy classes of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations is given by $(|\Delta_K(-1)|-1)/2$. Since the Alexander polynomial of the trefoil knot is t^2-t+1, we have one conjugacy class of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$-representations. By Proposition 5.3, we can take a representative ρ of this conjugacy class as follows:

$$\rho : \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_i \mapsto \begin{pmatrix} \zeta_3^i & 0 \\ 0 & \zeta_3^{-i} \end{pmatrix}$$

where $\zeta_3 = e^{2\pi\sqrt{-1}/3}$. The composition of ρ with the adjoint action is expressed as

$$(6) \qquad Ad \circ \rho(\mu) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Ad \circ \rho(x_i) = \begin{pmatrix} \zeta_3^{2i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3^{-2i} \end{pmatrix}$$

with respect to the basis $\{E, H, F\}$ of $\mathfrak{sl}_2(\mathbb{C})$ as in (1).

With $\alpha(\mu) = t$ and $\alpha(x_i) = 1$ in mind, we can express the twisted Alexander invariant as the following ratio of two determinants:

$$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = \frac{\det \left(\Phi_{Ad \circ \rho} \left(\dfrac{\partial r_i}{\partial x_j} \right) \right)_{\substack{1 \leqq i \leqq 2, \\ 1 \leqq j \leqq 2}}}{\det(\Phi_{Ad \circ \rho}(\mu - 1))}$$

where $r_1 = \mu x_1 \mu^{-1} x_2 x_1^{-1}$ and $r_2 = \mu x_2^{-1} x_1 \mu^{-1} x_2$ and $\partial r_i / \partial x_j$ is Fox differential of the word r_i by x_i.

The Fox differentials $\partial r_i / \partial x_j$ $(1 \leqq i, j \leqq 2)$ turn into

$$\begin{aligned} \frac{\partial r_1}{\partial x_1} &= \frac{\partial}{\partial x_1} \mu x_1 \mu^{-1} x_2 x_1^{-1} & \frac{\partial r_1}{\partial x_2} &= \frac{\partial}{\partial x_2} \mu x_1 \mu^{-1} x_2 x_1^{-1} \\ &= \mu - \mu x_1 \mu^{-1} x_2 x_1^{-1} & &= \mu x_1 \mu^{-1}, \\ &= \mu - 1, & & \\ \frac{\partial r_2}{\partial x_1} &= \frac{\partial}{\partial x_1} \mu x_2^{-1} x_1 \mu^{-1} x_2 & \frac{\partial r_2}{\partial x_2} &= \frac{\partial}{\partial x_2} \mu x_2^{-1} x_1 \mu^{-1} x_2 \\ &= \mu x_2^{-1}, & &= -\mu x_2^{-1} + \mu x_2^{-1} x_1 \mu^{-1} \\ & & &= -\mu x_2^{-1} + x_2^{-1}. \end{aligned}$$

Therefore the twisted Alexander invariant $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ turns out

$$\begin{aligned} \Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) &= \frac{\det \left(\Phi_{Ad \circ \rho} \left(\dfrac{\partial r_i}{\partial x_j} \right) \right)_{\substack{1 \leqq i \leqq 2, \\ 1 \leqq j \leqq 2}}}{\det(\Phi_{Ad \circ \rho}(\mu - 1))} \\ (7) \qquad &= \frac{\det \begin{pmatrix} \Phi_{Ad \circ \rho}(\mu - 1) & \Phi_{Ad \circ \rho}(\mu x_1 \mu^{-1}) \\ \Phi_{Ad \circ \rho}(\mu x_2^{-1}) & \Phi_{Ad \circ \rho}(-\mu x_2^{-1} + x_2^{-1}) \end{pmatrix}}{\det(\Phi_{Ad \circ \rho}(\mu - 1))} \end{aligned}$$

When we substitute (6) into the numerator and the denominator in (7), we have the determinant in the numerator:

$$(8) \quad \det \begin{pmatrix} -1 & 0 & -t & \zeta_3^{-2} & 0 & 0 \\ 0 & -t-1 & 0 & 0 & 1 & 0 \\ -t & 0 & -1 & 0 & 0 & \zeta_3^2 \\ 0 & 0 & -t\zeta_3^4 & \zeta_3^2 & 0 & t\zeta_3^4 \\ 0 & -t & 0 & 0 & t+1 & 0 \\ -t\zeta_3^{-4} & 0 & 0 & t\zeta_3^{-4} & 0 & \zeta_3^{-2} \end{pmatrix} = -t^6 - t^5 + t^4 + 2t^3 + t^2 - t - 1$$

and the determinant in the denominator:

$$\det\begin{pmatrix} -1 & 0 & -t \\ 0 & -t-1 & 0 \\ -t & 0 & -1 \end{pmatrix} = (t+1)(t^2-1). \tag{9}$$

By replacing the numerator and denominator in (7) with the determinants (8) & (9) and reducing this rational function, we have

$$\begin{aligned} \Delta_{E_K}^{\alpha\otimes Ad\circ\rho}(t) &= \frac{-t^6 - t^5 + t^4 + 2t^3 + t^2 - t - 1}{(t+1)(t^2-1)} \\ &= \frac{-(t-1)^2(t+1)^2(t^2+t+1)}{(t-1)(t+1)^2} \\ &= -(t-1)\Delta_K(-t). \end{aligned}$$

6.2. **The figure eight knot.** We consider the figure eight knot and the free Seifert surface illustrated as in Figure 9.

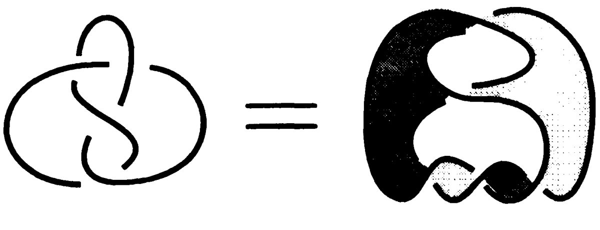

FIGURE 9. A free Seifert surface S of the figure eight knot

The spine of the Seifert surface S is a bouquet $a_1 \vee a_2$ of two circles and the closed loops corresponding to generators of $\pi_1(S^3 \setminus N(S))$ are illustrated as in Figure 10.

FIGURE 10. The spine of S and the generators x_1 and x_2 of $\pi_1(S^3 \setminus N(S))$

The Lin presentation associated with the Seifert surface S is expressed as

$$\begin{aligned} \pi_1(E_K) &= \langle x_1, x_2, \mu \mid \mu a_1^+ \mu^{-1} = a_1^-, \mu a_2^+ \mu^{-1} = a_2^- \rangle \\ &= \langle x_1, x_2, \mu \mid \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2 x_1 \mu^{-1} = x_2 \rangle. \end{aligned}$$

The number of conjugacy classes of irreducible metabelian $SL_2(\mathbb{C})$-representations is given by $(|\Delta_K(-1)|-1)/2$. Since $\Delta_K(t) = t^2 - 3t + 1$ for the figure eight knot, we have two conjugacy classes of irreducible metabelian $SL_2(\mathbb{C})$-representations. The representatives of these conjugacy classes is given by the following $SL_2(\mathbb{C})$-representations ρ_1 and ρ_2:

$$\rho_k : \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_1 \mapsto \begin{pmatrix} \zeta_5^k & 0 \\ 0 & \zeta_5^{-k} \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} \zeta_5^{2k} & 0 \\ 0 & \zeta_5^{-2k} \end{pmatrix} \quad (k = 1, 2)$$

where $\zeta_5 = e^{2\pi\sqrt{-1}/5}$. The composition of ρ_k with the adjoint action is expressed as

$$Ad \circ \rho_k(\mu) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Ad \circ \rho_k(x_i) = \begin{pmatrix} \zeta_5^{2ki} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_5^{-2ki} \end{pmatrix} \quad (i = 1, 2).$$

The twisted Alexander invariant for ρ_k is given by

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho_k}(t) = \frac{\det\left(\Phi_{Ad\circ\rho_k}\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1\leqq i\leqq 2,\\ 1\leqq j\leqq 2}}}{\det(\Phi_{Ad\circ\rho_k}(\mu - 1))}$$

where $r_1 = \mu x_1 \mu^{-1} x_2 x_1^{-1}$ and $r_2 = \mu x_2 x_1 \mu^{-1} x_2^{-1}$.

The Fox differentials $\partial r_i/\partial x_j$ $(1 \leqq i, j \leqq 2)$ turn into

$$\frac{\partial r_1}{\partial x_1} = \mu - \mu x_1 \mu^{-1} x_2 x_1^{-1}, \qquad \frac{\partial r_1}{\partial x_2} = \mu x_1 \mu^{-1}$$
$$= \mu - 1,$$
$$\frac{\partial r_2}{\partial x_1} = \mu x_2, \qquad \frac{\partial r_2}{\partial x_2} = \mu - \mu x_2 x_1 \mu^{-1} x_2^{-1}$$
$$= \mu - 1.$$

For ρ_1, the numerator of the twisted Alexander invariant is expressed as

$$\det\left(\Phi_{Ad\circ\rho_1}\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1\leqq i\leqq 2,\\ 1\leqq j\leqq 2}} = \det\begin{pmatrix} -1 & 0 & -t & \zeta_5^{-2} & 0 & 0 \\ 0 & -t-1 & 0 & 0 & 1 & 0 \\ -t & 0 & -1 & 0 & 0 & \zeta_5^2 \\ 0 & 0 & -t\zeta_5^{-4} & -1 & 0 & -t \\ 0 & -t & 0 & 0 & -t-1 & 0 \\ -t\zeta_5^4 & 0 & 0 & -t & 0 & -1 \end{pmatrix}$$
$$= (t^2 + 3t + 1)(t^4 - (\zeta_5^2 + \zeta_5 + \zeta_5^{-1} + \zeta_5^{-2} + 3)t^2 + 1)$$
$$= (t^2 + 3t + 1)(t^2 - 1)^2$$
$$= (t^2 + 3t + 1)(t - 1)^2(t + 1)^2$$

Since the denominator of the twisted Alexander invariant is given by $(t - 1)(t + 1)^2$ (see Eq. (9)), we have

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho_1}(t) = \frac{(t^2 + 3t + 1)(t - 1)^2(t + 1)^2}{(t - 1)(t + 1)^2}$$
$$= (t - 1)\Delta_K(-t).$$

For the other irreducible metabelian $SL_2(\mathbb{C})$-representation ρ_2, we have the same result as that for ρ_1.

6.3. 5_2 **knot.** Last we consider the 5_2 knot and the free Seifert surface illustrated as in Figure 11. This knot is often called a twist knot with type $(-2, 3)$. The trefoil knot, the figure eight knot and 5_2 are the first three non–trivial examples in twist knots. (We follows the convention of twist knots along [3, 4].)

The spine of the Seifert surface S is a bouquet $a_1 \vee a_2$ of two circles and the closed loops corresponding to generators of $\pi_1(S^3 \setminus N(S))$ are illustrated as in Figure 12.

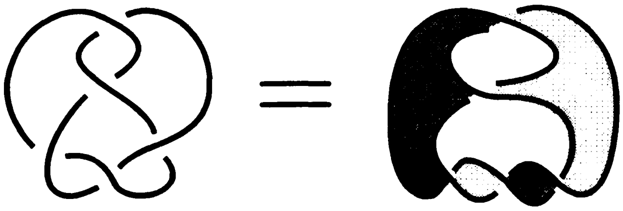

FIGURE 11. A free Seifert surface S of the 5_2 knot

FIGURE 12. The spine of S and the generators x_1 and x_2 of $\pi_1(S^3 \setminus N(S))$

The Lin presentation associated with the Seifert surface S is expressed as

$$\begin{aligned}\pi_1(E_K) &= \langle x_1, x_2, \mu \,|\, \mu a_1^+ \mu^{-1} = a_1^-, \mu a_2^+ \mu^{-1} = a_2^- \rangle \\ &= \langle x_1, x_2, \mu \,|\, \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2^{-2} x_1 \mu^{-1} = x_2^{-2} \rangle.\end{aligned}$$

The number of conjugacy classes of irreducible metabelian $SL_2(\mathbb{C})$-representations is given by $(|\Delta_K(-1)| - 1)/2$. Since $\Delta_K(t) = 2t^2 - 3t + 2$ for the 5_2 knot, we have three conjugacy classes of irreducible metabelian $SL_2(\mathbb{C})$-representations. The representatives of these conjugacy classes is given by the following $SL_2(\mathbb{C})$-representations ρ_k $(k = 1, 2, 3)$:

$$\rho_k : \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_1 \mapsto \begin{pmatrix} \zeta_7^k & 0 \\ 0 & \zeta_7^{-k} \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} \zeta_7^{2k} & 0 \\ 0 & \zeta_7^{-2k} \end{pmatrix}$$

where $\zeta_7 = e^{2\pi\sqrt{-1}/7}$. The composition of ρ_k with the adjoint action is expressed as

$$Ad \circ \rho_k(\mu) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Ad \circ \rho_k(x_i) = \begin{pmatrix} \zeta_7^{2ki} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_7^{-2ki} \end{pmatrix} \quad (i = 1, 2).$$

The twisted Alexander invariant for ρ_k is given by

$$\Delta_{E_K}^{\alpha \otimes Ad\circ\rho_k}(t) = \frac{\det\left(\Phi_{Ad\circ\rho_k}\left(\dfrac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1 \leqq i \leqq 2, \\ 1 \leqq j \leqq 2}}}{\det(\Phi_{Ad\circ\rho_k}(\mu - 1))}$$

where $r_1 = \mu x_1 \mu^{-1} x_2 x_1^{-1}$ and $r_2 = \mu x_2^{-2} x_1 \mu^{-1} x_2^2$.

The Fox differentials $\partial r_i/\partial x_j$ $(1 \leqq i, j \leqq 2)$ turn into

$$\frac{\partial r_1}{\partial x_1} = \mu - \mu x_1 \mu^{-1} x_2 x_1^{-1}, \qquad \frac{\partial r_1}{\partial x_2} = \mu x_1 \mu^{-1}$$
$$= \mu - 1,$$
$$\frac{\partial r_2}{\partial x_1} = \mu x_2^{-2}, \qquad \frac{\partial r_2}{\partial x_2} = -\mu x_2^{-1} - \mu x_2^{-2} + \mu x_2^{-2} x_1 \mu^{-1} + \mu x_2^{-2} x_1 \mu^{-1} x_2$$
$$= -\mu x_2^{-1} - \mu x_2^{-2} + x_2^{-2} + x_2^{-1}.$$

For ρ_1, the numerator of the twisted Alexander invariant is expressed as

$$\det\left(\Phi_{Ad\circ\rho_1}\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{1 \leqq i,j \leqq 2}$$
$$= \det\begin{pmatrix} -1 & 0 & -t & \zeta_7^{-2} & 0 & 0 \\ 0 & -t-1 & 0 & 0 & 1 & 0 \\ -t & 0 & -1 & 0 & 0 & \zeta_7^2 \\ 0 & 0 & -t\zeta_7^8 & \zeta_7^{10}+\zeta_7^6 & 0 & t\zeta_7^8 + t\zeta_7^4 \\ 0 & -t & 0 & 0 & 2t+2 & 0 \\ -t\zeta_7^{-8} & 0 & 0 & t\zeta_7^{-8}+t\zeta_7^{-4} & 0 & \zeta_7^{-10}+\zeta_7^{-6} \end{pmatrix}$$
$$= (2t^2+3t+2)$$
$$\cdot(-(\zeta_7^3 + \zeta_7^{-3} + 2)t^4 + (\zeta_7^3 - \zeta_7^2 - \zeta_7 - \zeta_7^{-1} - \zeta_7^{-2} + \zeta_7^{-3} + 3)t^2 - \zeta_7^3 - \zeta_7^{-3} - 2)$$
$$= -(2t^2+3t+2)(\zeta_7^3 + \zeta_7^{-3} + 2)(t^2-1)^2$$
$$= -(2t^2+3t+2)(\zeta_7^3 + \zeta_7^{-3} + 2)(t-1)^2(t+1)^2.$$

Since the denominator of the twisted Alexander invariant is given by $(t-1)(t+1)^2$ (see Eq. (9)), we have

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho_1}(t) = \frac{-(\zeta_7^3 + \zeta_7^{-3} + 2)(2t^2+3t+2)(t-1)^2(t+1)^2}{(t-1)(t+1)^2}$$
$$= -(\zeta_7^3 + \zeta_7^{-3} + 2)(t-1)\Delta_K(-t).$$

Similarly, we have the twisted Alexander invariants for ρ_2 and ρ_3 as follows:

$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho_2}(t) = -(\zeta_7 + \zeta_7 + 2)(t-1)\Delta_K(-t),$$
$$\Delta_{E_K}^{\alpha\otimes Ad\circ\rho_3}(t) = -(\zeta_7^2 + \zeta_7^{-2} + 2)(t-1)\Delta_K(-t).$$

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